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Operator improvement for Ginsparg-Wilson fermions

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Abstract

The improvement of fermionic operators for Ginsparg-Wilson fermions is investigated. We present explicit formulae for improved Green's functions, which apply both on-shell and off-shell.

1 Introduction

Recently, there has been a great deal of interest in Ginsparg-Wilson fermions [1], because it is now realised that they allow the calculation of chirally symmetric physics without having any doubling problem [2, 3, 4]. We also know that because of its chiral properties the Ginsparg-Wilson fermion matrix is automatically an $O(a)$ improved action [5], in exactly the same sense that the clover action is an improved action.

If we are interested in going beyond spectrum calculations to compute improved matrix elements, for example for structure functions and decay constants, we also need to know how to improve fermion operators. When we compute forward hadronic matrix elements, it is enough if the operators are improved for on-shell quantities. However, some methods of doing non-perturbative renormalisation [6] require calculations of off-shell Green's functions, with a virtuality large enough that we can reasonably compare with continuum perturbation theory. To do this well, we would like to be able to remove $O(a)$ effects from off-shell Green's functions too.

In this paper we find several ways of improving operators for the Ginsparg-Wilson action, both for on-shell and off-shell Green's functions. The paper is organised as follows. In sect. 2 we first study the fermion propagator, and in sect. 3 we test the results in the free case for a particular realisation of Ginsparg-Wilson fermions. In sect. 4 we then turn to our main subject, operator improvement. The

major outcome is that, in the simplest form, the improved operators do not require any coefficients to be tuned non-perturbatively. In sect. 5 we briefly comment on the Ward identities, and in sect. 6 we conclude.

2 Fermion propagator

In this section we review some known results concerning the Ginsparg-Wilson propagator, as preparation for our consideration of bilinear fermionic operators. The basic Ginsparg-Wilson condition is [1]

$$D_{GW}\,\gamma_5 + \gamma_5\,D_{GW} = a\,D_{GW}\,\gamma_5\,D_{GW}, \tag{1}$$

where D_{GW} is the Ginsparg-Wilson fermion matrix. The fermion matrix does not anti-commute with γ_5, but there is nevertheless a form of chiral symmetry present [4].

Let us now look more closely at the Ginsparg-Wilson matrix. From the matrix D_{GW} we can define a related matrix [7]

$$K_{GW} \equiv \left(1 - \frac{a}{2}D_{GW}\right)^{-1} D_{GW}. \tag{2}$$

We will discuss later what happens when D_{GW} has an eigenvalue equal to $2/a$. This implies

$$D_{GW} = \left(1 + \frac{a}{2}K_{GW}\right)^{-1} K_{GW}. \tag{3}$$

The eigenvalues of D_{GW} lie on a circle of radius $1/a$ and centre $1/a$, while the eigenvalues of K_{GW} lie on the imaginary axis. The relationship between the eigenvalues of the two matrices is shown in Fig. 1. If we substitute eq. (3) into the condition (1) we find that

$$K_{GW}\,\gamma_5 + \gamma_5\,K_{GW} = 0. \tag{4}$$

The chiral properties of K_{GW} are even closer to those of the continuum Dirac operator than D_{GW}. The fermion propagator we really want to use is the propagator calculated with K_{GW}. The propagator calculated from D_{GW} has contact terms of $O(a)$, which mean that if we Fourier transform it we will find $O(a)$ lattice artifacts when we are off-shell. A propagator calculated from K_{GW} contains no contact terms: it satisfies chirality even at zero distance. Therefore it will be improved off-shell too.

We want to write down improved Green's functions for the massive as well as for the massless theory. When we consider massive fermions, we will use the fermion matrix [5]

$$M \equiv \left(1 - \frac{1}{2}am_0\right) D_{GW} + m_0\,. \tag{5}$$

This is the usual choice when working with Ginsparg-Wilson fermions, and it has the advantage of giving a simple linear relationship between bare and renormalised fermion masses. Most other ways of adding

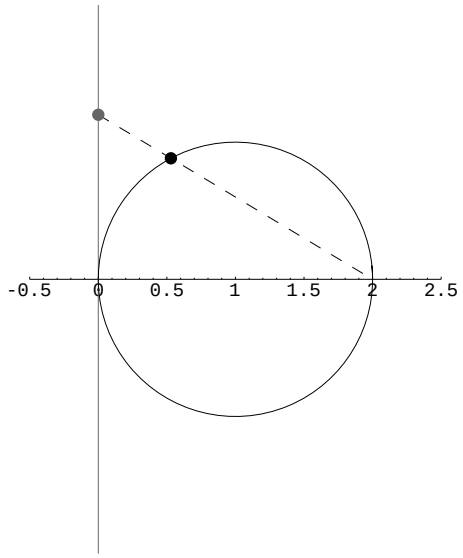

Figure 1: The spectrum of the Ginsparg-Wilson matrix D_{GW} (the circle) and of the associated matrix K_{GW} (dotted vertical line). The dashed line links an eigenvalue of D_{GW} to the corresponding eigenvalue of K_{GW}.

a quark mass lead to non-linear relationships of the form $m_R \propto m_0(1 + am_0 b_m + \cdots)$. However, despite the slightly more complicated form of the answer, the improvement schemes we suggest in this paper also work with minor modifications for other choices of M, for example $M = D_{GW} + m_0$.

The fermion propagator

$$S \equiv \frac{1}{a^4} \left\langle M^{-1} \right\rangle \tag{6}$$

still contains contact terms violating chiral symmetry.[1] The fermion propagator with the correct chiral symmetry properties is the propagator corresponding to K_{GW}, i.e.

$$S_\star = \frac{1}{a^4} \left\langle \frac{1}{K_{GW} + m_0} \right\rangle . \tag{7}$$

We will always use $\star$ to denote improved quantities. The improved Green's functions have the correct chiral properties, and we therefore know that they are free of $O(a)$ discretisation errors. This is because any $O(a)$ term in a Green's function has the opposite chirality to the leading term, so checking chirality is a simple way of testing for $O(a)$ discretisation errors. The improved propagator (7) can be calculated

[1] We will be considering Green's functions involving charged or coloured fermions. These are, of course, only defined if the gauge is fixed in some way, so expectation values are always to be understood as averages over gauge configurations with some gauge-fixing term present. Which gauge is chosen makes no difference to any of the results in this paper, as all the improvement coefficients are gauge independent.

in terms of the fermion matrix M. Inserting the definition (2) into eq. (7) we find

$$S_\star(x,y) = \frac{1}{1+am_0b_\psi}\left(S(x,y) - \frac{a}{2}\lambda_\psi\delta(x-y)\right), \tag{8}$$

with improvement coefficients

$$b_\psi = -\frac{1}{2}, \tag{9}$$

$$\lambda_\psi = 1. \tag{10}$$

We use the lattice delta function

$$\delta(x-y) \equiv \frac{1}{a^4}\delta_{xy}, \tag{11}$$

where δ_{xy} is the dimensionless Kronecker delta function. Although the propagators S and $S_\star$ in eq. (8) are of course gauge-dependent, the improvement coefficients b_ψ and λ_ψ are gauge independent.

Note that although we have used K_{GW} at intermediate stages in the discussion, our final result (8) only involves inverting the fermion matrix M. As noted before, K_{GW} is not well defined if D_{GW} has eigenvalues exactly equal to $2/a$, which will happen in topologically non-trivial configurations. On the other hand, the fermion matrix M has no such problems. It is always invertible for $m_0 > 0$, so eq. (8) can be applied even in configurations with a non-trivial winding number. Our improved Green's functions will always be of such a form that only M^{-1} appears as a propagator. An identity equivalent to eq. (8) that will often prove useful is

$$\frac{1}{K_{GW}+m_0} = \left(1-\frac{a}{2}D_{GW}\right)M^{-1} = M^{-1}\left(1-\frac{a}{2}D_{GW}\right). \tag{12}$$

Now that we have defined a propagator, we can look for an expression for the chiral condensate. The natural choice of order parameter for chiral symmetry is to take the trace of the improved propagator $S_\star$,

$$\left\langle\bar\psi(x)\psi(x)\right\rangle_\star = \mathrm{Tr}\, S_\star(x,x) = \frac{1}{a^4}\mathrm{Tr}\left\langle\left(1-\frac{a}{2}D_{GW}\right)M^{-1}\right\rangle. \tag{13}$$

The final form of the improved chiral condensate is the same as that given in [8].

3 The fermion propagator in the free theory

In this section we consider one explicit realisation of the Ginsparg-Wilson condition (1), namely Neuberger's Dirac operator [3]. We show that the formulae we derived do indeed lead to results free of $O(a)$ effects.

Starting from the massless Wilson fermion matrix D_W, Neuberger introduces the matrix A, defined by

$$A \equiv 1 - aD_W. \tag{14}$$

It can then be shown that the operator

$$D_N \equiv \frac{1}{a}\left(1 - A/\sqrt{A^\dagger A}\right) \tag{15}$$

satisfies the Ginsparg-Wilson condition (1) .

In the free theory, the Wilson matrix is diagonal in momentum, and has the value

$$D_W(p) = \mathrm{i}\not{s} + W, \tag{16}$$

where we use

$$W = \sum_\mu (1 - \cos ap_\mu), \tag{17}$$

$$\not{s} = \sum_\mu \gamma_\mu \sin ap_\mu, \tag{18}$$

$$s^2 = \sum_\mu \sin^2 ap_\mu. \tag{19}$$

Calculating D_N, the Ginsparg-Wilson matrix corresponding to D_W, we find [4]

$$A^\dagger A = (1 - W)^2 + s^2 \tag{20}$$

and

$$D_N(p) = \frac{1}{a}\,\frac{\mathrm{i}\not{s} + \left(W - 1 + \sqrt{(1-W)^2 + s^2}\right)}{\sqrt{(1-W)^2 + s^2}}. \tag{21}$$

If we expand $D_N(p)$ for small p, we find

$$D_N(p) = \mathrm{i}\not{p} + \frac{1}{2}ap^2 + O(a^2p^3), \tag{22}$$

so D_N still has discretisation errors of $O(a)$.

However, when we calculate $K_N(p)$ according to the formula (2), we find

$$K_N(p) = \frac{2}{a}\,\frac{\mathrm{i}\not{s}}{1 - W + \sqrt{(1-W)^2 + s^2}}. \tag{23}$$

Finally in $K_N(p)$ we have an operator which anti-commutes with γ_5. When we expand it for small p, we find

$$K_N(p) = \frac{\mathrm{i}\sum_\mu \gamma_\mu (p_\mu - \frac{1}{6}a^2 p_\mu^3 + O(a^4p^5))}{1 - \frac{1}{4}a^2p^2 + O(a^4p^4)} \tag{24}$$

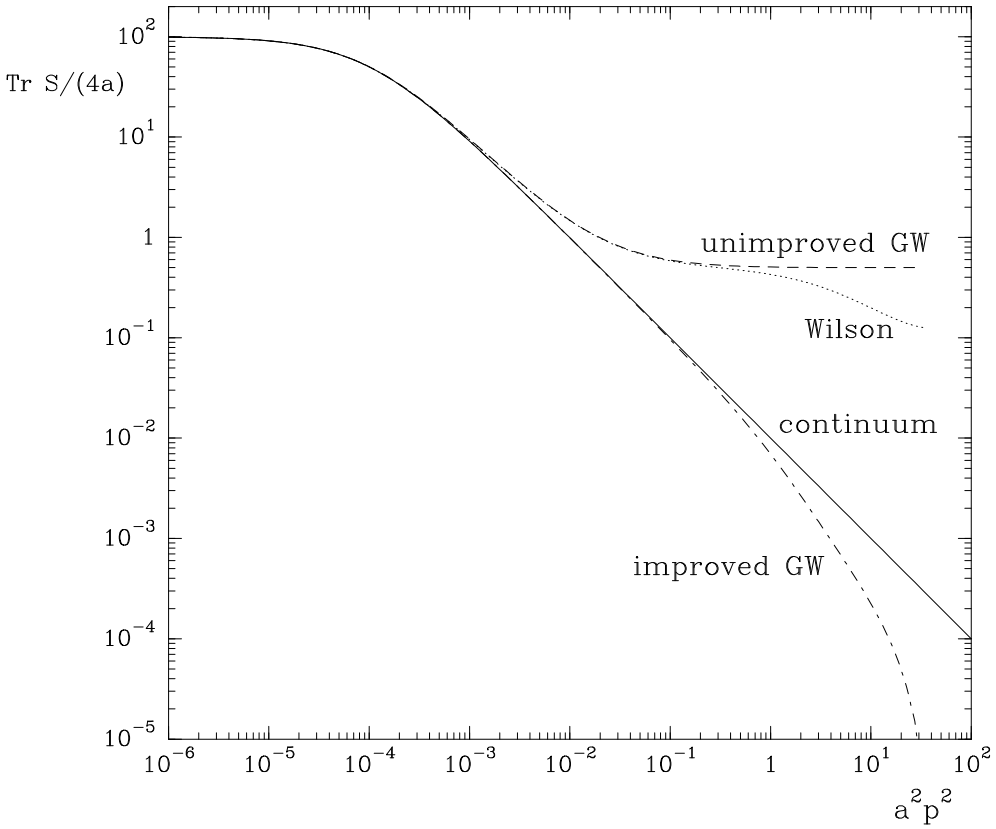

Figure 2: The scalar part of the free propagator, $(1/4a)\mathrm{Tr}S(p)$, plotted against a^2p^2, for the various fermion matrices considered in this section. The Wilson propagator is the dotted line, the unimproved propagator from D_N is the dashed line, the improved propagator $S_\star$ is the dot-dashed line, and the solid line is the continuum result $m_0/(p^2+m_0^2)$. All curves are plotted for $am_0 = 0.01$. The momentum is taken in the direction $(1,1,1,1)$.

which has lattice errors of $O(a^2)$.

However, K_N does have problems of its own. It diverges when $s^2 = 0$ and $W \geq 1$, which occurs at the 'doubler' momenta $pa = (\pi,0,0,0),\cdots$. Because it has poles in momentum space, it would also be an extremely non-local matrix if Fourier transformed back into position space. This is, of course, inevitable: the Nielsen-Ninomiya theorem tells us that a fermion matrix that anti-commutes with γ_5 and has no doubling must be non-local.

The improved propagator does not have these difficulties. It is

$$S_\star(p,m_0) \;=\; \frac{-\mathrm{i}\not{s}/a + m_0\left(1-W+\sqrt{(1-W)^2+s^2}\right)/2}{2\left(W-1+\sqrt{(1-W)^2+s^2}\right)/a^2 + m_0^2\left(1-W+\sqrt{(1-W)^2+s^2}\right)/2}$$

$$= \frac{-\mathrm{i}\not{p} + m_0}{p^2 + m_0^2} + O(a^2). \tag{25}$$

The denominator has no zeroes if $m_0^2 > 0$, and if $m_0 = 0$ the only zero is at $p = 0$. The numerator vanishes at the 'doubler' momenta. If we expand this propagator as a power series in a, we see that there is no $O(a)$ term: the discretisation errors of $S_\star$ are of $O(a^2)$.

In Fig. 2 we compare the trace of the various lattice propagators with the correct continuum result. The Wilson propagator and the unimproved Ginsparg-Wilson propagator M^{-1} both deviate from the correct result when $a^2p^2 \approx am_0$, but the improved propagator $S_\star$ is good up to $a^2p^2 \approx 1$.

4 Improvement of flavour non-singlet fermion operators

Next we want to improve the Green's function corresponding to a flavour non-singlet operator

$$\mathcal{O} = \bar{\psi} O \psi, \tag{26}$$

where O can include Dirac structure and covariant derivatives. Covariant derivatives can be represented by the usual formulae

$$\begin{aligned} \overrightarrow{D}_\mu\, f &= \frac{1}{2a}\left[U_\mu(x) f(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu}) f(x-\hat{\mu})\right], \\ \bar{f}\, \overleftarrow{D}_\mu &= \frac{1}{2a}\left[\bar{f}(x+\hat{\mu}) U_\mu^\dagger(x) - \bar{f}(x-\hat{\mu}) U_\mu(x-\hat{\mu})\right], \end{aligned} \tag{27}$$

or by any other expression with discretisation errors of $O(a^2)$. We want our improved Green's function to be

$$G_\star^{\mathcal{O}} = \frac{1}{a^4}\left\langle \frac{1}{K_{GW} + m_0} O \frac{1}{K_{GW} + m_0} \right\rangle. \tag{28}$$

This will have discretisation errors of $O(a^2)$, because the propagators used have been improved, and the usual discretisation of O is valid up to terms of $O(a^2)$.

We need a formula analogous to eq. (8), giving $G_\star^{\mathcal{O}}$ in terms of M^{-1}. The simplest way to construct such an identity is by using eq. (12), which gives

$$G_\star^{\mathcal{O}} = \frac{1}{a^4} \frac{1}{1 + am_0 b_\psi} \left\langle M^{-1} \widetilde{O} M^{-1} \right\rangle, \tag{29}$$

where

$$\begin{aligned} \widetilde{O} &= (1 + am_0 b_\psi)\left(1 - \frac{a}{2} D_{GW}\right) O \left(1 - \frac{a}{2} D_{GW}\right) \\ &= O + am_0 b_\psi O - \frac{a}{2}(1 + am_0 b_\psi)(D_{GW} O + O\, D_{GW}) + \frac{a^2}{4}(1 + am_0 b_\psi) D_{GW} O\, D_{GW}. \end{aligned} \tag{30}$$

Equation (29) is written with the same wave function improvement factor as eq. (8), because the quantity of physical interest is always the ratio $G_\star/S_\star$, and we want the wave function factors to cancel.

Equation (29) is not the most general expression for $G_\star^{\mathcal{O}}$. We can use the equation of motion

$$D_{GW}M^{-1} = -\frac{m_0}{1-am_0/2}M^{-1} + \frac{1}{1-am_0/2}\delta_{xy} \tag{31}$$

to show that this is equivalent to the expression

$$G_\star^{\mathcal{O}} = \frac{1}{1+am_0b_\psi}\left[G_\circ - \frac{a}{2}\lambda_{\mathcal{O}}C^{\mathcal{O}} + \frac{a^2}{4}\eta_{\mathcal{O}}\langle O\rangle\right], \tag{32}$$

where

$$G_\circ \equiv \left\langle M^{-1}O_\star M^{-1}\right\rangle, \tag{33}$$

$$O_\star \equiv O + am_0c_0O - \frac{a}{2}c_1(D_{GW}O + O\,D_{GW}) + \frac{a^2}{4}c_2D_{GW}O\,D_{GW}, \tag{34}$$

$$C^{\mathcal{O}} \equiv \left\langle OM^{-1}\right\rangle + \left\langle M^{-1}O\right\rangle, \tag{35}$$

with

$$c_0 = \frac{\frac{1}{2}-c_1}{1-\frac{1}{2}am_0} - \frac{am_0c_2}{4(1-\frac{1}{2}am_0)^2}, \tag{36}$$

$$\lambda_{\mathcal{O}} = \frac{1-c_1}{1-\frac{1}{2}am_0} - \frac{am_0c_2}{2(1-\frac{1}{2}am_0)^2}, \tag{37}$$

$$\eta_{\mathcal{O}} = \frac{1-c_2-\frac{1}{2}am_0}{(1-\frac{1}{2}am_0)^2}. \tag{38}$$

There are two free parameters in this system of equations. The improvement coefficients c_1 and c_2 can take any value, but once it is chosen, the values of the other improvement coefficients are fixed. This freedom comes from the equations of motion, which allow us to compensate for a change in one of the improvement coefficients by adjusting the other coefficients. For an example of this in the clover action see [9].

The terms proportional to $C^{\mathcal{O}}$ and $\langle O\rangle$ can be interpreted as contact terms. The Green's function we are interested in has the form $\langle\psi_i\bar{\psi}_jO_{jk}\psi_k\bar{\psi}_l\rangle$. On the lattice we should expect to see a contact term of the form $\delta_{ij}\langle O_{jk}\psi_k\bar{\psi}_l\rangle + \langle\psi_i\bar{\psi}_jO_{jk}\rangle\delta_{kl}$, with a coefficient of $O(a)$, and a 'double contact term' of the form $\delta_{ij}\langle O_{jk}\rangle\delta_{kl}$, with a coefficient of $O(a^2)$.

If we are looking at the operator Green's function on-shell, these contact terms are irrelevant, because the fermion fields will all be well separated in position, and so the delta functions are all zero. In this case we can use any values of c_1 and c_2, as long as we use the correct c_0 value.

On the other hand, if we look at the off-shell Green's functions, we have to take the contact terms into account. One possibility would be that as well as computing $G_\circ$ one would also compute $C^{\mathcal{O}}$ and $\langle O\rangle$ and add them with the coefficients λ_O and η_O. A more elegant procedure would be to use eq. (29),

i.e. to choose c_1 and c_2 so that the contact terms are absent, and $G_\circ$ is improved both on- and off-shell. The improvement coefficients for this special case are

$$\begin{aligned} c_1 = c_2 &= 1 - \frac{1}{2} a m_0, \\ c_0 &= -\frac{1}{2}. \end{aligned} \tag{39}$$

Note that, as in the previous section, the matrix K_{GW} is only used heuristically. To find the improved Green's functions $S_\star$ and $G^{\mathcal{O}}_\star$, the only matrix that we really have to invert is M.

Finally, we mention another way to calculate the improved Green's functions $S_\star$ and $G^{\mathcal{O}}_\star$ of eqs. (8), (28). From eq. (12) we can write the improved propagator as

$$S_\star = \left\langle \left(1 - \frac{a}{2} D_{GW}\right) M^{-1} \right\rangle . \tag{40}$$

Since D_{GW} and M commute, this can be rewritten in the more symmetric form

$$S_\star = \left\langle \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} M^{-1} \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} \right\rangle . \tag{41}$$

This formula gives exactly the same improved propagator as eq. (8), but it lends itself to a somewhat different interpretation. In eq. (8) we subtract an unwanted contact term present in the unimproved propagator, while eq. (41) can be interpreted as the propagator of a 'rotated' fermion field, which is similar to the picture of clover improvement presented in [10].

Similar formulae can be written for the improved Green's functions, where instead of adding irrelevant higher dimension terms to O we perform a 'rotation' of the fermion fields. The resulting formula is

$$G^{\mathcal{O}}_\star = \left\langle \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} M^{-1} O^{\mathrm{rot}} M^{-1} \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} \right\rangle , \tag{42}$$

where

$$O^{\mathrm{rot}} \equiv \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} O \left(1 - \frac{a}{2} D_{GW}\right)^{\frac{1}{2}} . \tag{43}$$

Using the identity (12) it is easy to see that eq. (42) is equivalent to eq. (28). The rotation approach requires us to take the square root of a matrix, which is not needed in the irrelevant operator approach, so it might be more costly to implement.

As with the propagator, we can illustrate the effects of improving the Green's function by looking at a simple case in the free theory, namely the Green's function $G^\mu(p, m_0)$ for the local vector current $\bar{\psi}\gamma_\mu\tau\psi$. Here τ is a flavour matrix, $\mathrm{Tr}\,\tau = 0$, normalised so that $\mathrm{Tr}\,\tau^2 = 1$. As in the case of the propagator, the most sensitive test is to look at the scalar part of the Green's function, which is shown in Fig. 3. We have chosen the quantity

$$\sum_\mu \mathrm{i} \frac{\sin a p_\mu}{a} \frac{1}{4a^2} \mathrm{Tr}\left[G^\mu(p, m_0)\,\tau\right] , \tag{44}$$

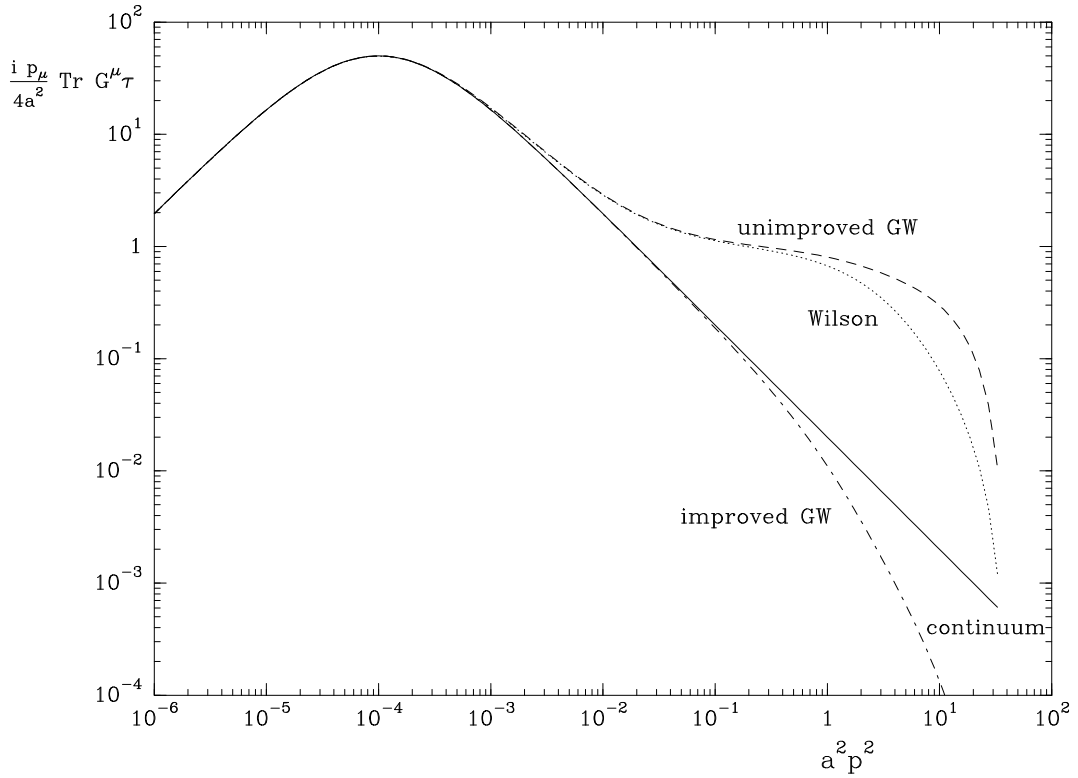

Figure 3: The scalar part of the Green's function for the local vector current $ip_\mu/(4a^2)\mathrm{Tr}[G^\mu(p)\tau]$. The Wilson fermion gives the dotted line, the result from Ginsparg-Wilson fermions with no improvement terms is the dashed line, the dot-dashed line is from the improved Green's function $G_\star$, and the solid line is the continuum result $2m_0p^2/(p^2+m_0^2)^2$. As in Fig. 2 all curves are plotted for $am_0 = 0.01$, and the momentum is taken in the direction $(1,1,1,1)$.

which has the value $2m_0p^2/(p^2+m_0^2)^2$ in the continuum. Again, the Wilson action and the unimproved Ginsparg-Wilson Green's function deviate significantly from the desired continuum result at $a^2p^2 \approx am_0$, while the improved Green's function only has errors of $O(a^2)$ and remains reliable until $a^2p^2 \approx 1$.

As a final example in the free case, let us take an operator with a derivative, namely

$$O^{\mu\nu} \equiv \frac{\mathrm{i}}{2}\bar{\psi}\,\frac{1}{2}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\tau\,\psi, \tag{45}$$

with $\mu \neq \nu$. This is an operator that can be used to compute the flavour non-singlet moment $\langle x \rangle$ of a hadronic structure function. In the continuum, the free Green's function for this operator has the value

$$G^{\mu\nu}_{\mathrm{cont}}(p,m_0) = \frac{-\mathrm{i}\not{p}+m_0}{p^2+m_0^2}\,\frac{1}{2}\left(\gamma_\mu p_\nu + \gamma_\nu p_\mu\right)\,\tau\,\frac{-\mathrm{i}\not{p}+m_0}{p^2+m_0^2}. \tag{46}$$

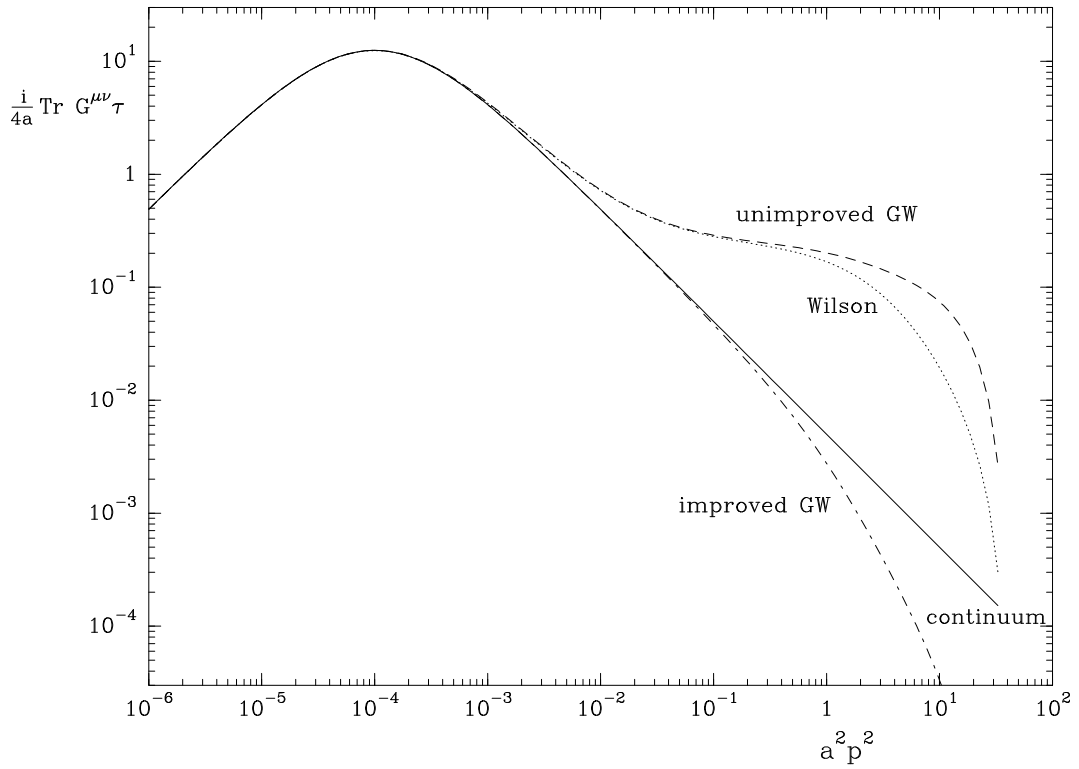

Figure 4: The scalar part of the Green's function for the operator $O^{\mu\nu}$ of eq. (45), $\mathrm{i}/(4a)\mathrm{Tr}[G^{\mu\nu}(p,m_0)\tau]$. The Wilson fermion gives the dotted line, the result from Ginsparg-Wilson fermions with no improvement terms is the dashed line, the dot-dashed line is from the improved Green's function $G_\star$, and the solid line is the continuum result $2m_0p^\mu p^\nu/(p^2+m_0^2)^2$. As in Fig.2 all curves are plotted for $am_0 = 0.01$ and the momentum is taken in the direction $(1,1,1,1)$.

The improved lattice Green's function found by applying the formulae in sect. 4 is

$$G^{\mu\nu}_\star(p,m_0) = S_\star(p,m_0)\,\frac{1}{2a}\,(\gamma_\mu \sin ap_\nu + \gamma_\nu \sin ap_\mu)\;\tau\, S_\star(p,m_0), \tag{47}$$

where the explicit form of $S_\star(p,m_0)$ is given in eq. (25). Since the discretisation errors of $S_\star$ are $O(a^2)$ it is clear that the difference between $G^{\mu\nu}_{\rm cont}(p,m_0)$ and $G^{\mu\nu}_\star(p,m_0)$ is also of $O(a^2)$. This is illustrated in Fig. 4 , where we plot the trace of $G^{\mu\nu}$.

5 Ward Identities

In the continuum chiral symmetry relates the scalar part of the fermion propagator to the forward Green's function for the non-singlet pseudoscalar operator $\bar{\psi}\gamma_5\tau\psi$:

$$\gamma_5\tau S(p) + S(p)\gamma_5\tau = 2mG^5(p)\,. \tag{48}$$

On the lattice we can check that the improved Green's functions of sect. 4 satisfy the same Ward identity:

$$\gamma_5\tau S_\star(p) + S_\star(p)\gamma_5\tau = 2m_0G^5_\star(p)\,. \tag{49}$$

One might naively expect that this Ward identity would be violated at $O(a^2)$ by the discretisation errors, but in fact it holds exactly, even though both the left hand side and the right hand side have discretisation errors of $O(a^2)$.

Note that all the Green's functions needed for Ward identities, like eq. (49), can be computed on the lattice. So we can hope to use them to determine the improvement coefficients in other improved theories, such as clover fermions, where we cannot calculate them analytically from first principles.

6 Conclusions

We see that using the Ginsparg-Wilson action there are many ways of constructing propagators and Green's functions which are correct to $O(a^2)$, both on-shell and off-shell. In this paper we have considered both, improvement by adding the Green's functions of irrelevant operators, and by rotating the fermion fields.

The idea is very simple. In eqs. (7) and (28) we have the Green's functions we wish to obtain, with the correct chiral symmetries, and therefore free from all $O(a)$ errors. However, there they are expressed in terms of the associated matrix K_{GW}, defined in eq. (2). Fortunately, it is easy to find equivalent expressions in terms of the well-defined matrix M, so we never need to find or invert K_{GW} explicitly.

The improvement coefficients are universal. It makes no difference which theory is being considered, Abelian or non-Abelian gauge theory, or a theory with no gauge symmetry, or what value of the coupling is being used. It does not even matter which operator is being considered. Operator improvement for other improved actions, such as clover fermions, will certainly not be so simple. The improvement coefficients in that case will certainly be functions of the coupling, and depend on the theory considered. Nevertheless, Ginsparg-Wilson fermions may give some hints as to what sort of improvement terms are needed, and what sort of contact terms are to be expected.

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